

A Beginning Approach to The Theory of Generalized Modular Forms – ”A-Castle Forms”, Automorphic Factors, etc.

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May 2025

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Abstract

In this paper, we will present a few new extensions of the theory of modular forms, a departure from the typical theory – i.e. the theory of modular forms over the matrix group $SL_2(\mathbb{Z})$ with inputs in $\mathcal{H}_{\mathbb{C}}$.

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1 Introduction

1.1 Introductory Theory – Clarification

A modular form, in the standard literature, M , of weight k and level Γ is a holomorphic function from the complex upper half plane, $\mathcal{H}$, to the complex numbers, $\mathbb{C}$, which satisfies the following conditions:

- 1: $\forall \gamma \in \Gamma \subset SL_2(\mathbb{Z}), f(\gamma(z)) = (cz + d)^k f(z)$.
- 2: $\forall \gamma \in SL_2(\mathbb{Z}), (cz + d)^{-k} f(\gamma(z))$ is bounded across the limit $\lim_{z \rightarrow i\infty}$.

2 Results of Research

2.1 Modular Forms in $SL_k(\mathbb{Z})$

The first topic we are to go over is defining modular forms w.r.t. "generalized Möbius transformations" emerging from matrices in $SL_k(\mathbb{Z})$ for a general $k \in \mathbb{Z}^+$. We introduce the notion of an "A-castle form" for a subgroup χ of $SL_k(\mathbb{Z})$, where an A-castle form of weight w and level χ is a "modular form-like" function based off of a subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$, A .

For our A-castle forms, they follow an analogous set of conditions:

$$1: \text{ For all } \frac{T_{g_1}}{T_{g_2}} \in \chi \subset SL_k(\mathbb{Z}), f\left(\frac{T_{g_1}(z)}{T_{g_2}(z)}\right) = T_{g_2}^w f(z).$$

$$2: \text{ For all } \frac{T_{g_1}}{T_{g_2}} \in \chi \subset SL_k(\mathbb{Z}), T_{g_2}^{-w} f(\gamma(z)) \text{ is bounded across the limit } \lim_{z \rightarrow i\infty}.$$

For a matrix, M , in $SL_k(\mathbb{Z})$, we define $T_i(z) = \sum_{j=0}^k a_{ij} z^{j-1}$. We use the term "generalized Möbius transformation" to refer to the rational function of two such $T, \frac{T_{l_1}}{T_{l_2}}$ for $l_1, l_2 \in \{1, \dots, k\}$.

For "standard" modular forms where $k=2$, we obtain our usual equations as discussed in 1.1 when we consider a so-called A-castle form using the singleton set $A = \{(1, 2)\}$, constituting the transformation $\frac{T_1(z)}{T_2(z)}$, where $T_1 = az + b, T_2 =$

$cz + d$, given a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This is trivial to verify by simple substitution.

There are 2 properties that modular forms satisfy, namely those of addition and multiplication:

(1): $M + N$ is a modular form whenever M and N are;

(2): MN is a modular form, also whenever M and N are.

We will now prove that the 2 identities above hold for A-castle forms as well.

We will construct these 2 proofs similarly to how one would construct the proofs of (1) and (2).

Proof of (1).

I.a: G_1, G_2 are A-castle forms, for the same A, χ and weight w .

II.a: This means $G_1(\gamma(z)) = D(\gamma)^w G_1(z)$, and $G_2(\gamma(z)) = D(\gamma(z))^w G_2(z)$.

III.a: Invoking the distributive property, we get $(G_1 + G_2)(\gamma(z)) = D(\gamma)^w (G_1 + G_2)(z)$.

IV.a: Thus, the first transformation property is satisfied.

I.b: Similar to subproof a, we have $G_1 + G_2$ satisfying the second condition -- the boundedness condition -- as noted in 1.1. This is since a sum of 2 bounded functions is a bounded function itself. This is trivial to prove.

Conclusion: The sum of 2 A-castle forms of the same weight is an A-castle form.

Proof of (2).

I.a: Similar to before, G_1 and G_2 both satisfy the first condition.

Additionally, G_1 has a weight of k , and G_2 has a weight of l .

II.a: $(G_1 G_2)(\gamma(z)) = D(\gamma)^k D(\gamma)^l \cdot z = D^{k+l}(\gamma) \cdot z$.

I.b: The product of 2 bounded functions is necessarily a bounded function itself. This is trivial to prove.

Conclusion: The product of 2 A-castle forms is an A-castle form. Additionally, just like "regular" modular forms, A-castle forms create a monoid under addition and multiplication. This is because they also satisfy the other 2 monoid axioms, which are associativity and identity.

2.2 "Generalized Automorphic Factors"

Previously, we discussed A-castle forms. Now, we will move onto the theory of "automorphic factors", or, functions of the type:

$$\varepsilon(a, b, c, d)(cz + d)^k$$

defined w.r.t. $SL_2(\mathbb{R})$. Which, are proposed to generalize the condition of modularity. Here, we call ε the "nebenotypus" of the automorphic factor. We can replace the $(cz + d)^k$ factor in the original condition to give:

$$\forall \gamma \in SL_2(\mathbb{Z}), f(\gamma(z)) = \varepsilon(a, b, c, d)(cz + d)^k f(z)$$

Now, when we have an A-castle form of weight w and level χ , G , in $SL_n(\mathbb{Z})$ we can create a similar condition with a "generalized" nebenotypus, ψ , which is the following:

$$\forall \frac{T_{g_1}}{T_{g_2}} = T_a \in H_\chi \subset SL_k(\mathbb{Z}), G(T(z)) = \psi(a_{11} \dots a_{kk}) T_{g_2}^{-w} G(z)$$

An automorphic factor itself, is a function ν which is analytic and can be written in the form of a product of a function v and an integer power of the denominator of a transformation γ :

$$\nu(\gamma, z) = v(\gamma)(cz + d)^k$$

where $|v(\gamma)| = 1$.

The way above is one possible way to generalize the theory of automorphic factors, and is what it would seem the "most intuitive" for the theory we've built up.

2.3 Eisenstein Series

The Eisenstein series for a positive even integer, d , is a special, interesting case of a family of modular forms, in which this paper, deserves its own section.

The typical definition of the Eisenstein series, G_d is $G_d(\tau) = \sum_{m,n \in \mathbb{Z} - \{0\}} \frac{1}{(m+n\tau)^m}$, $\tau \in \mathcal{H}_0^{CD}$. Keep in mind, this is for $\frac{d}{2} > 1$.

We will now cover the first way we can generalize this particular family of modular forms. We can rewrite the definition as $G_d(s) = \sum_{m\omega_1 + n\omega_2 = \lambda \in \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2} \frac{1}{(m\omega_1 + \frac{1}{i}m\omega_2)^s}$.

When $\Lambda = \mathbb{Z} + \mathbb{Z}i$, or, $\Lambda = \mathbb{Z}[i]$, it reduces to the usual definition of the Eisenstein series of "weight" d shown above. We can easily verify this by having $\omega_1 = 1, \omega_2 = i$ in our definition, and noting $\frac{\omega_2}{i} = 1$. The second way we can

generalize this family of modular forms is by considering larger lattices, e.g. lattices over $\mathbb{H}$ or $\mathbb{O}$. We define the following "Generalized Eisenstein series", for a lattice, Φ , in $CD_m(\mathbb{C})$ which is generated by the 2^{m+1} -ions $\omega_1, \dots, \omega_{2^{m+1}}$, which the function will be of several $\tau_i \in CD_m(\mathbb{C})$ –

$$G_d^\Phi(\tau_1 \dots \tau_{2^{m+1}-1}) = \sum_{p_1, \dots, p_{2^{m+1}} \in \mathbb{Z} - \{0\}} \frac{1}{\left(\sum_{j=1}^{2^{m+1}} p_j \omega_j^\Phi \begin{cases} j \geq 2 & \frac{\tau_j - 1}{i_{m+1,j}} \\ j = 1 & 1 \end{cases} \right)^d}$$

Here, analogously it reduces to a "normal" Eisenstein function when our lattice is equal to $\mathbb{Z} + \mathbb{Z}i_{m+1,1} + \dots + \mathbb{Z}i_{m+1,2^m-1}$, and it further reduces to the Eisenstein series we first defined in this section if $m=0$, which is the typical Eisenstein series known in the literature.

To quickly clarify our notation, we use the convention $i_{n,k}$ as the n^{th} imaginary unit of $CD_{k-1}(\mathbb{C})$.

We are aware, in the standard literature in the theory of elliptic functions and modular forms of the mapping for certain lattices, $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \rightarrow \mathbb{Z} + \mathbb{Z}\tau$, $\tau = \frac{\omega_2}{\omega_1}$. However, a direct analogy of this convenient mapping is not possible for larger lattices, as there is no way currently known of to condense $\omega_1, \dots, \omega_{2^m}$ into a single parameter that can describe any lattice (of a certain type) in the way τ for (certain) complex lattices does.

2.4 Generalizations of the congruence subgroup $\Gamma_0(N)$

The Hecke congruence subgroup, $\Gamma_0(N)$ is usually constructed as the subgroup of $SL_2(\mathbb{Z})$, where $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \% N = I_{2 \times 2} \right\}$, where $M \% k_{ij} = M_{ij} \% k$, $\%$ being our notation for "reduced modulo". We then have the congruence subgroup $\Gamma_1(N)$, which we similarly define by the constraint where $a, b \% N = 1, c \% N = 0$. We see here the constraint has been relaxed on b . We can create a "generalized" congruence group that is "like" $\Gamma_0(N)$ through the following construction: We have a positive integer, d . That is, an element of $\mathbb{Z}^+$. Now, we consider the multiplicative group of rank d tensors, or, tensors with d indices used to specify their elements. For example, matrices have 2 indices used to specify each element, so we consider them "rank 2 tensors". We introduce the notation $\Gamma_0^d(\mathbb{Z})$, which denotes the group: $\Gamma_{0,m}^d(\mathbb{Z}) = \{A \in SL_{\underbrace{m \times \dots \times m}_d}(\mathbb{Z}) \mid A_{\underbrace{i \dots i}_d} \% B = 1, A_{i_1, \dots, i_d} \% N = 0 \text{ given } \neg(i_1 = \dots = i_d)\}$. This

works as well as our initial family of subgroups of $SL_2(\mathbb{Z})$, $\Gamma_0(N)$ if we consider the concepts of hyperdeterminant and tensor multiplication, but we cannot be easily assured that the group axioms are even satisfied given a product of 2 hypermatrices, or, higher dimensional tensors.

We can also note the existence of the family of groups

$$\Gamma_0^{r \times r}(N) = \left\{ A = \begin{pmatrix} a_{11} & \cdot & \cdot & a_{r1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{1r} & \cdot & \cdot & a_{rr} \end{pmatrix} \in SL_r(\mathbb{Z}) : A \% N = I_{r \times r} \right\}, r \in \mathbb{Z}^+$$

, for which we can construct "congruence" A -castle forms.

These are easy to construct, but it's unclear how we would construct the $\Gamma_1^{r \times r}(N)$ counterpart to this family. For now, we will construct the group by relaxing the constraint on a_{r1} , as we did in the case of the "original" $\Gamma_1(N)$ group, where we relaxed the constraint on the "top right" element b .

2.5 "Hypercomplex Modular Forms"

Here, basing our insight off the typical definition of a modular form, we can generalize the domain of modular forms to any upper half-plane in any hypercomplex number system, i.e. a Cayley-Dickson construction.

We let $\mathcal{H}_n^{CD}$ be the "upper half plane" of the n^{th} Cayley-Dickson construction from $\mathbb{C}$ onwards, where $\mathcal{H}_0^{CD} \triangleq \mathcal{H}_{\mathbb{C}}$. That is, the subset, H , of $CD_{n-1}(\mathbb{C})$ of that such construction s.t. $\forall p \in H, \text{sgn}(\Im_{2^n-1}(p)) = 1$. We can obtain this set from $CD_{n-1}(\mathbb{C})$ by applying the transformation:

$$a_0 + a_1 i_1 + \dots + a_{2^n-1} i_{2^n-1} \rightarrow \begin{cases} a_0 + \dots + a_{2^n-1} i_{2^n-1} & \text{sgn}(a_{2^n-1}) = 1 \\ a_0 + \dots - a_{2^n-1} i_{2^n-1} & \text{sgn}(a_{2^n-1}) = -1 \end{cases}$$

. This acts as a "quotient map" which maps the entire space onto its upper half plane. We can similarly obtain such a half plane from $\mathbb{R}^{2^n}$, which is naturally isomorphic to $CD_{n-1}(\mathbb{C})$. We have the standard q -series, where $q = e^{2\pi i n z}$. We can then substitute z for any p in $\mathcal{H}_k^{CD}$, to obtain a generalized "q-series" which will help us define modular forms in this manner.

Keep in mind, we will have to compute the terms with the Taylor series of e^x , which, for computerized applications, may be cause for very volatile computational complexity, especially if we want a decent amount of precision (10 or so terms). Since hypercomplex numbers transform similarly to $\mathbb{C}$, it is expected that they too can generate modular forms, yet we have yet to prove/disprove the affirmative. A formal treatment of this conjecture will be given in a later paper.

3 Works Referenced/Utilized/Further Reading

Robert Rankin, *Modular Forms and Functions* (1977) – Section 3, pp. 70-88
 Enrico De Ronche – *Matrix valued Hilbert modular forms* (further reading: related work)